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SOME STUDY ON SOLITARY TRAVELING WAVE SOLUTIONS FOR NONLINEAR EVOLUTION EQUATIONS

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Abstract: This article aims to study the exact solitary wave solutions of some Nonlinear Evolution Equations (NLEEs) by using a novel expansion method. Three nonlinear models of physical significance i.e. Pochhammer-Chree Equations (PCE) equation, Newell-Whitehead Equation(NWE) and Fitzhugh-Nagumo Equation(FNE) are considered. The method that we have chosen, is simple, straightforward and, gives the three types of solutions including trigonometric, exponential, and rational solutions as compared to other existing methods. Various types of periodic and solitary wave solutions are derived.
Keywords: Solitons, NLEEs, Newell-Whitehead Equation(NWE), Soliton, Fitzhugh-Nagumo Equation(FNE), Pochhammer-Chree Equations (PCE).
Mathematics Subject Classification: 35Qxx, 35C08, 35L05

1 Introduction

In sciences, engineering, biological sciences, and ocean sciences many phenomena are arising such as rouge waves formation, multicellular biological dynamics, population dynamics, and shallow water waves that are nonlinear in nature. Waves dynamics of these dynamical systems can be arranged by a group of integrable nonlinear partial differential equations(NLPDEs) which can be expressed in terms of Nonlinear Evolution Equations (NLEEs) of integer orders. Therefore, study of such NLEEs describing ocean waves and their corresponding solitary wave solutions, which are widely known as soliton of NLEEs [1, 2, 3, 4, 5, 6, 7, 8, 9], reveals a compelling stint in the investigations of behavior of nonlinearity in the ocean sciences. Some of the important approaches for solving these types of NLEEs, like the Hirota method [1], Jacobi elliptic function solutions [2], the exponential expansion method [3], the ansatz [4], the Rational form method [5], the sine-cosine expansion method [6], the Kudryashov method [7], rationalisation method [8], the homogenous balance method [9], the sub-equation method [10], the exp-function method [11], the truncated Painlevé expansion [12] and many others [13] are widely used, but to deal with the NLEEs non of the above methods are universal in character that can be used . In the last two decades, Wang *et al.* [14] have popularized a approach called $(\frac{G'}{G})$-expansion method for a appropriate investigation of NLEEs in waves dynamics. Subsequently applications of this formalism has also been communicated in literature [15, 16, 17, 18, 19, 20, 21, 22, 23] in different fields of wave dynamics. In this work, we study few NLEEs of physical models which has its own importance in wave sciences. We have focused only on three NLEEs,

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first the Pochhammer-Chree Equations (PCE) which is modeled for propagation of longitudinal deformation waves [24]. Second model is the Fitzhugh-Nagumo Equation(FNE) which used as a model for reaction-diffusion equation (RD), noise formations in circuit theory, evolution of waves etc. [25]. The third models is Newell-Whitehead Equation(NWE), which models the intercommunication of the development of the diffusion term with the nonlinear repercussion of the reaction denomination [26, 27]. We tried to find some new solutions, that to best of our knowledge are not done earlier.

2 The Formalism

Let us assume a general nonlinear NLEE as

$$R(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{yy}, \ldots) = 0. \quad (1)$$

where $u = u(x, y, t)$, in which derivatives of highest order along with nonlinear terms are convoluted. Introducing the wave transformations $\xi = (x + y - ct)$, changes $u(x, y, t) = u(\xi)$. Based on this, eq.(1) in form of NLEE changes to an ODE as

$$P(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \ldots) = 0. \quad (2)$$

The formalism suggest us to assume the solution of eq.(2) as a series i.e.

$$u(\xi) = \alpha_m(\frac{G'}{G})^m + \alpha_{m-1}(\frac{G'}{G})^{m-1} + \ldots., \quad (3)$$

Here $G = G(\xi)$ assures the setting

$$G'' + \tau G' + \kappa G = 0, \quad (4)$$

where α_m, α_{m-1},.....,α_0, τ and κ are constants to be dogged later and $\alpha_m \neq 0$. Formalism suggests us to make a compatible balance between nonlinear terms appearing in ODE and the highest order derivatives in eq.(2), to find the integer m. Exchanging eq.(3) with eq.(2) and using eq.(4), further rationalising the order of $(\frac{G'}{G})$, and then comparing each coefficient to zero will bring in a set of equations involving α_m, α_{m-1},.....,α_0, c, τ and κ. Substituting α_m, α_{m-1},.....,α_0 and c and the accustomed solutions of eq. (4) into eq.(3), we extract further some new traveling wave solutions of NLEEs (1). Further depending on the sign of the discriminant $\triangle = \tau^2 - 4\kappa$ and with Eq.(4) possess the three types of general solutions as

$$(\frac{G\prime}{G}) = \begin{cases} \frac{\sqrt{\tau^2-4\kappa}}{2}\left[\frac{A\sinh(\frac{\sqrt{\tau^2-4\kappa}}{2})\chi + B\cosh(\frac{\sqrt{\tau^2-4\kappa}}{2})\chi}{A\cosh(\frac{\sqrt{\tau^2-4\kappa}}{2})\chi + B\sinh(\frac{\sqrt{\tau^2-4\kappa}}{2})\chi}\right] - \frac{\kappa}{2}, & \text{if } \triangle > 0 \\ \frac{\sqrt{4\kappa-\tau^2}}{2}\left[\frac{-A\sinh(\frac{\sqrt{4\kappa-\tau^2}}{2})\chi + B\cosh(\frac{\sqrt{4\kappa-\tau^2}}{2})\chi}{A\cosh(\frac{\sqrt{4\kappa-\tau^2}}{2})\chi + B\sinh(\frac{\sqrt{4\kappa-\tau^2}}{2})\chi}\right] - \frac{\kappa}{2}, & \text{if } \triangle < 0 \\ \frac{B}{A+B\chi} - \frac{\kappa}{2} & \text{if } \triangle = 0 \end{cases}$$

or more simplified version as

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$$(\frac{G'}{G}) = \begin{cases} \frac{\sqrt{\tau^2-4\kappa}}{2}\tanh(\frac{\sqrt{\tau^2-4\kappa}}{2}\chi + \chi_0) - \frac{\kappa}{2}, & \text{if } \triangle > 0, \tanh\chi_0 = \frac{A}{B}, \left|\frac{A}{B}\right| > 1 \\ \frac{\sqrt{\tau^2-4\kappa}}{2}\coth(\frac{\sqrt{\tau^2-4\kappa}}{2}\chi + \chi_0) - \frac{\kappa}{2}, & \text{if } \triangle > 0, \coth\chi_0 = \frac{A}{B}, \left|\frac{A}{B}\right| < 1 \\ \frac{\sqrt{4\kappa-\tau^2}}{2}\cot(\frac{\sqrt{4\kappa-\tau^2}}{2}\chi + \chi_0) - \frac{\kappa}{2}, & \text{if } \triangle > 0, \cot\chi_0 = \frac{A}{B}, \left|\frac{B}{A}\right| < 1 \\ \frac{B}{A+B\chi} - \frac{\kappa}{2}. & \text{if } \triangle = 0. \end{cases}$$

With the recipe of the $(\frac{G'}{G})$-expansion method, we try to solve three PDEs of physical importance as discussed.

3 The Pochhammer-Chree Equations (PCE)

Here, we study the PCE, which reads as

$$u_{tt} - u_{ttxx} - (\alpha u + \beta u^{n+1} + \gamma u^{2n+1})_{xx} = 0, \quad n > 1. \tag{5}$$

Earlier this was solved by Balubasky [?] for $n = 1$ and $n = 2$. Ideally, we investigate the PCE when $n = 1$ by using $(\frac{G'}{G})$-formalism.

Case-1: For $n = 1$, and $\xi = x + y - ct$, eq. (5) converts to

$$c^2u'' + (c^2 - \alpha)u - \beta u^2 - \gamma u^3 = 0. \tag{6}$$

Making a compatible correspondence between u'' and u^3 in eq.(6), we get, $m = 1$. Solution of eq.(6), as given by formalism is of the form

$$u(\xi) = \alpha_1(\frac{G'}{G}) + \alpha_0, \quad \alpha_1 \neq 0. \tag{7}$$

Capitalising eq.(7) into eq.(6) and rationalising power of $(\frac{G'}{G})$ in sync, converted into the polynomials in $(\frac{G'}{G})$. Further making coefficients of polynomials to zero, leads a set of algebraic equations:

$$2\alpha_1 c^2 + \gamma\alpha_1^3 = 0,$$

$$3c^2\tau\alpha_1 + 3\gamma\alpha_1^2\alpha_0 + \beta\alpha_1^2 = 0,$$

$$[(\tau^2 + 2\kappa)c^2 - (c^2 - \alpha)]\alpha_1 + 3\gamma\alpha_1\alpha_0^2 + 2\beta\alpha_1\alpha_0 = 0,$$

$$c^2\kappa\tau\alpha_1 + \gamma\alpha_0^3 + \beta\alpha_0^2 - (c^2 - \alpha)\alpha_0 = 0. \tag{8}$$

figuring out the equations, we find

$$\alpha_1 = \sqrt{\frac{-2}{\gamma}}c, \quad \alpha_0 = -\frac{(3\tau c + \beta\sqrt{\frac{-2}{\gamma}})}{3\sqrt{-2\gamma}}, \quad c = \pm\sqrt{\frac{2\beta^2 + 9\alpha\gamma}{9\gamma}},$$

$$\alpha_1 = -\sqrt{\frac{-2}{\gamma}}c, \quad \alpha_0 = \frac{(3\tau c - \beta\sqrt{\frac{-2}{\gamma}})}{3\sqrt{-2\gamma}}, \quad c = \pm\sqrt{\frac{9\alpha\gamma - 2\beta^2}{9\gamma}}. \tag{9}$$

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τ, κ, α, β and γ are arbitrary constants. Now, substituting eq.(9) into eq.(7) we get the solution of eq.(5) as follows:

$$u_1(\xi) = \sqrt{\frac{-2}{\gamma}}c(\frac{G'}{G}) - \frac{(3\tau c+\beta\sqrt{\frac{-2}{\gamma}})}{3\sqrt{-2\gamma}},$$

$$u_2(\xi) = -\sqrt{\frac{-2}{\gamma}}c(\frac{G'}{G}) + \frac{(3\tau c-\beta\sqrt{\frac{-2}{\gamma}})}{3\sqrt{-2\gamma}}. \quad (10)$$

We have three category of travelling wave solutions of eq.(5) as follows:

When $\triangle = \tau^2 - 4\kappa > 0$,

$$u_1(\xi) = \frac{1}{2}\sqrt{\frac{-2}{\gamma}}c\sqrt{\tau^2-4\kappa}(\frac{A\sinh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi+B\cosh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi}{A\cosh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi+B\sinh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi}) - \frac{\beta}{3\gamma}. \quad (11)$$

$$u_2(\xi) = -\frac{1}{2}\sqrt{\frac{-2}{\gamma}}c\sqrt{\tau^2-4\kappa}(\frac{A\sinh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi+B\cosh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi}{A\cosh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi+B\sinh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi}) - \frac{\beta}{3\gamma}. \quad (12)$$

where A and B are arbitrary constants. In particular, if $A \neq 0$, B=0, $\tau > 0$, $\kappa = 0$, then u_1 and u_2 becomes

$$u_1(\xi) = \frac{1}{2}\tau c\sqrt{\frac{-2}{\gamma}}\tanh\frac{\tau}{2}\xi - \frac{\beta}{3\gamma}, \quad (13)$$

$$u_2(\xi) = -\frac{1}{2}\tau c\sqrt{\frac{-2}{\gamma}}\tanh\frac{\tau}{2}\xi - \frac{\beta}{3\gamma}. \quad (14)$$

when $\triangle =< 0$,

$$u_1(\xi) = \frac{1}{2}\sqrt{\frac{-2}{\gamma}}c\sqrt{4\kappa-\tau^2}(\frac{-A\sin(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi+B\cos(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi}{A\cos(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi+B\sin(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi}) - \frac{\beta}{3\gamma}. \quad (15)$$

$$u_2(\xi) = -\frac{1}{2}\sqrt{\frac{-2}{\gamma}}c\sqrt{4\kappa-\tau^2}(\frac{-A\sin(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi+B\cos(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi}{A\cos(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi+B\sin(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi}) - \frac{\beta}{3\gamma}. \quad (16)$$

when $\triangle = 0$,

$$u_1(\xi) = \sqrt{\frac{-2}{\gamma}}c(\frac{B}{A+B\xi}) - \frac{\beta}{3\gamma}, \quad (17)$$

$$u_2(\xi) = -\sqrt{\frac{-2}{\gamma}}c(\frac{B}{A+B\xi}) - \frac{\beta}{3\gamma}. \quad (18)$$

These could be the soliton solutions of the PCE (when $n = 1$) under different parametric conditions.

Case-2: Here, we study the PCE for α=0, $\beta = -\frac{1}{2}$ and γ=0 which was successfully done earlier by [?]. The PCE for $n = 1$ is

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$$u_{tt} - u_{ttxx} - (-\frac{u^2}{2})_{xx} = 0. \quad (19)$$

using $\xi = x + y - ct$, (19) is switched to an ODE

$$c^2u'' - c^2u'''' - (-\frac{1}{2}u^2)'' = 0, \quad (20)$$

Integrating eq.(20) twice

$$-c^2u'' + \frac{1}{2}u^2 + c^2u = 0. \quad (21)$$

Making a compatible balance we get, $m = 2$. Solution of eq.(21) is of the form

$$u(\xi) = \alpha_2(\frac{G'}{G})^2 + \alpha_1(\frac{G'}{G}) + \alpha_0, \quad \alpha_2 \neq 0, \quad (22)$$

following same process as described, we get simultaneous equations as follows:

$$-6c^2\alpha_2 + \frac{1}{2}\alpha_2^2 = 0,$$
$$-(2\alpha_1 + 10\tau\alpha_2)c^2 + \alpha_1\alpha_2 = 0,$$
$$-(8\kappa\alpha_1 + 3\tau\alpha_1 + 4\tau^2\alpha_2)c^2 + \frac{1}{2}(\alpha_1^2 + 2\alpha_2\alpha_0) + c^2\alpha_2 = 0,$$
$$-(6\kappa\tau\alpha_2 + 3\kappa\alpha_1 + \tau^2\alpha_1)c^2 + \alpha_1\alpha_0 + c^2\alpha_1 = 0,$$
$$-(2\kappa^2\alpha_2 + \kappa\tau\alpha_1)c^2 + \frac{1}{2}\alpha_0^2 + c^2\alpha_0 = 0. \quad (23)$$

On solving these simultaneous equations, we get

$$\alpha_2 = 12c^2, \ \alpha_1 = 12\tau c^2, \alpha_0 = 0, \ \tau^2 + 8\kappa - 1 = 0. \quad (24)$$

Now, substituting eq. (24) into eq.(22) we get solution of eq.(19) as follows:

$$u(\xi) = 12c^2(\frac{G'}{G})^2 + 12\tau c^2(\frac{G'}{G}). \quad (25)$$

The method suggest, we have three types of solutions of eq.(19)

$$u_1(\xi) = 3c^2(\tau^2 - 4\kappa)(\frac{A\sinh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi + B\cosh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi}{A\cosh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi + B\sinh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi})^2 - 3c^2\tau^2, \quad (26)$$

In particular, if $A \neq 0$, B=0, $\tau > 0$, $\kappa = 0$, then u_1 becomes

$$u_1(\xi) = 3c^2\tau^2\tanh^2\frac{\tau}{2}\xi - 3c^2\tau^2. \quad (27)$$

Second

$$u_2(\xi) = 3c^2(4\kappa - \tau^2)(\frac{-A\sin(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi + B\cos(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi}{A\cos(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi + B\sin(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi})^2 - 3c^2\tau^2. \quad (28)$$

In particular, if $A \neq 0$, B=0, $\tau > 0$, $\kappa = 0$, then u becomes

$$u_2(\xi) = 3c^2\kappa^2\coth^2\frac{\kappa}{2}\xi - 3c^2\kappa^2. \quad (29)$$

which are the periodic or solitary wave solution of (19) Third

$$u_3(\xi) = 12c^2(\frac{B}{A+B\xi})^2 - 3c^2\tau^2, \quad (30)$$

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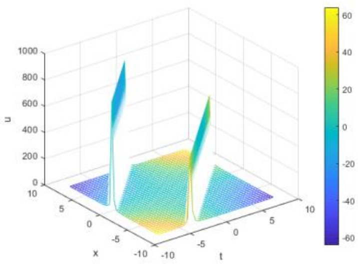
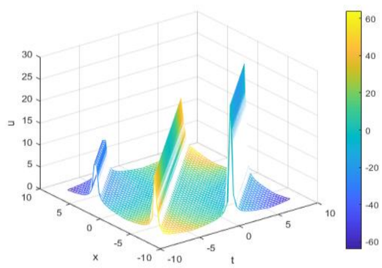
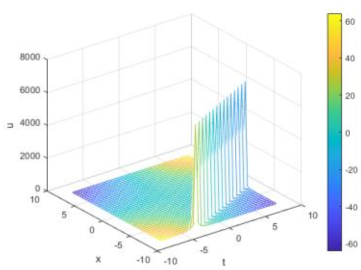
Figure 1: Traveling wave solution corresponding to the PC equation



Plot of $u_1(\zeta)$, when $\triangle > 0, \tau = 0.1, \kappa = 0.02 A = 0.02, B = 0.03$



Plot of $u_2(\zeta)$, when $\triangle < 0, \tau = 0.2, \kappa = 0.2, A = 0.02, B = 0.03$



Plot of $u_3(\zeta)$, when $\triangle = 0, \tau = 0.2, \kappa = 0.01, A = 0.02, B = 0.03$

Solutions are presented and profiles are plotted.

4 Newell-Whitehead Equation

The NWE Equation elaborates the dynamical performance near the bifurcation mark of the Rayleigh-Benard convection of binary fluid adulteration. Rayleigh-Benard convection is one of the widely studied convection development because of its interpretive and experimental convenience. It is because of the interaction of the diffusion term affect with the effect of nonlinear reaction term is modeled by it. This reads as

$$u_t = u_{xx} + au - bu^3. \tag{31}$$

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here a, b are rational constants. If $a=-4, b=-4$ and $n=3$, Then eq.(31) becomes the Allen-Cahn equation. Further eq.(31) is switched to

$$u''+cu'+au-bu^3=0. \qquad (32)$$

By making a balance between u'' with u^3 in eq.(32), we get $m=1$. So the solutions are

$$u(\xi)=\alpha_1(\frac{G'}{G})+\alpha_0, \quad \alpha_1 \neq 0. \qquad (33)$$

Following the same recipe, equating each coefficient polynomials of $\frac{G'}{G}$. to zero, yields:

$$\begin{aligned}
&2\alpha_1-b\alpha_1^3=0,\\
&(3\tau-c)\alpha_1-3b\alpha_1^2\alpha_0=0,\\
&(2\kappa+\tau^2-c\tau+a)\alpha_1-3b\alpha_1\alpha_0^2=0,\\
&(\tau-c)\kappa\alpha_1+a\alpha_0-b\alpha_0^3=0.
\end{aligned} \qquad (34)$$

On solving these, equations we have

$$\alpha_1=\pm\sqrt{\frac{2}{b}}, \quad \alpha_0=\pm\frac{(3\tau-c)}{3\sqrt{2b}}, \quad c=\pm3\sqrt{\frac{a}{2}}. \qquad (35)$$

Now, substituting eq.(35) into eq.(33) we get the solution of eq.(31) as follows:

$$u(\xi)=\pm\sqrt{\frac{2}{b}}(\frac{G'}{G})\pm\frac{(3\tau-c)}{3\sqrt{2b}}, \qquad (36)$$

Substituting the general solution of LODE into the formalism eq.(36), we have wave solutions of eq.(31) are as follows:

$$u(\xi)=\pm\sqrt{\frac{\tau^2-4\kappa}{2b}}\left(\frac{A\sinh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi+B\cosh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi}{A\cosh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi+B\sinh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi}\right)-\frac{1}{2}\sqrt{\frac{a}{b}}, \qquad (37)$$

further, if $A\neq0$, B=0, $\tau>0$, $\kappa=0$, leads to

$$u_1(\xi)=\pm\frac{\tau}{\sqrt{2b}}\tanh\frac{\tau}{2}\xi-\frac{1}{2}\sqrt{\frac{a}{b}}. \qquad (38)$$

$$u_2(\xi)=\pm\sqrt{\frac{4\kappa-\tau^2}{2b}}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi+B\cos(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi}{A\cos(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi+B\sin(\frac{1}{2}\sqrt{4\kappa-\tau^2})\xi}\right)-\frac{1}{2}\frac{\sqrt{a}}{\sqrt{b}}, \qquad (39)$$

$$u_3(\xi)=\pm\sqrt{\frac{2}{b}}\left(\frac{B}{A+B\xi}\right). \qquad (40)$$

These are traveling wave solutions of the NWE.

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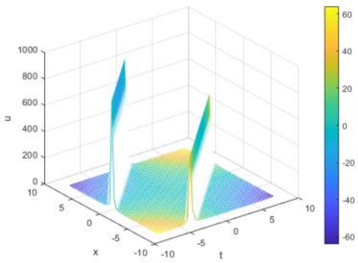
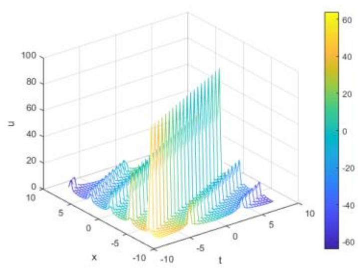
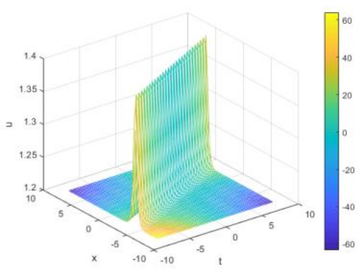
Figure 2: Traveling wave solution corresponding to the NW equation



Plot of $u_1(\zeta)$, when $\triangle > 0, \tau = 0.1, \kappa = 0.02, A = 0.02, B = 0.03$



Plot of $u_2(\zeta)$, when $\triangle < 0, \tau = 0.2, \kappa = 0.2, A = 0.02, B = 0.03$



Plot of $u_3(\zeta)$, when $\triangle = 0, \tau = 0.2, \kappa = 0.01, A = 0.02, B = 0.03$.

5 Fitzhugh-Nagumo Equation

The FitzHugh-Nagumo equations have been used to qualitatively model many biological phenomena. FNE reads as

$$u_t = u_{xx} - u(1-u)(\rho - u). \tag{41}$$

where $0 < \rho < 1$ and is of the unknown function depending on the temporal variable t and the spatial variable x. When, $\rho = -1$ reduces to the real Newell-Whitehead equation. Further eq.(41) can be switched to

$$u'' + cu' - u + 2u^2 - u^3 = 0. \tag{42}$$

Comparing highest order derivatives and nonlinear terms appearing in ODE, we get, $m = 1$. So

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as per the formalism solutions would be.

$$u(\xi) = \alpha_1(\frac{G'}{G}) + \alpha_0, \quad \alpha_1 \neq 0. \tag{43}$$

By using eq.(43) into eq.(42) and following same recipe yields a set of simultaneous algebraic equations and on solving these, we get

$$\alpha_1 = \sqrt{2}, \quad \alpha_0 = \frac{3\tau - c + 2\sqrt{2}}{3\sqrt{2}}, \quad c = -\sqrt{2}, \frac{1}{\sqrt{2}}, \tag{44}$$

$$\alpha_1 = -\sqrt{2}, \quad \alpha_0 = \frac{2\sqrt{2} - 3\tau + c}{3\sqrt{2}}, \quad c = \sqrt{2}, \frac{-1}{\sqrt{2}}. \tag{45}$$

Travelling wave solutions of eq.(42) are as follows:

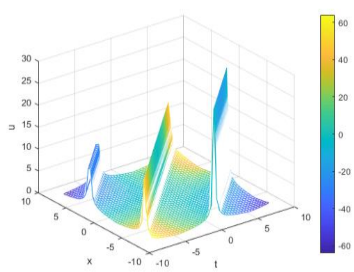
$$u_1(\xi) = \pm\sqrt{\frac{\tau^2 - 4\kappa}{2}}\left(\frac{A\sinh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi + B\cosh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi}{A\cosh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi + B\sinh(\frac{1}{2}\sqrt{\tau^2-4\kappa})\xi}\right) + \frac{1}{2}. \tag{46}$$

Here $\xi = x \pm \frac{1}{\sqrt{2}}t$ respectively, In particular, if $A \neq 0$, B=0, $\tau > 0$, $\kappa = 0$, leads to

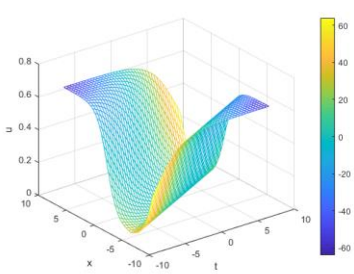
$$u_1(\xi) = \pm\frac{\tau}{\sqrt{2}}\tanh\frac{\tau}{2}\xi + \frac{1}{2}. \tag{47}$$

$$u_2(\xi) = \pm\sqrt{2}(\frac{B}{A + B\xi}) + 1. \tag{48}$$

Figure 3: Traveling wave solution corresponding to the Fitzhugh-Nagumo equation



Plot of $u_1(\chi)$, when $\triangle > 0, \gamma = 0.1, \rho = 0.02, A = 0.02, B = 0.03$



Plot of $u_2(\chi)$, when $\triangle = 0, \gamma = 0.2, \rho = 0.01, A = 0.01, B = 0.02$
Solutions of FNE are presented and profiles are plotted.

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6 Conclusion

It is worthwhile to mention that these hyperbolic, trigonometric and rational form of solutions cannot be obtained by any of the methods mentioned in the introduction. The proposed method is sententious, direct, simplified and this can be used further for many other NLEEs in the future. This approach gives us solitary wave solutions under certain parametric restrictions as seen from the profiles. Results of NWE, PCE equation reflects the multi-soliton solutions in their first kind which is visible in their plots and surface distribution graph for $\tau = 0.2, \kappa = 0.2, A = 0.02, B = 0.03$. This solution of PCE are in coherence with the results obtained by [24] under certain parametric restrictions. Further it is observed that solutions of third kind leads to a single soliton solutions except FNE which shows a cusp like solutions for $\triangle = 0, \tau = 0.2, \kappa = 0.01, A = 0.02, B = 0.03$. Solutions are rich in structures, however in ours plots we may have taken specified values for simplicity, but this may not change the nature of the profile in different parametric conditions. This work predicts that, this $(\frac{G\prime}{G})$ method is quite methodical and experientially well studied for finding solitary solutions for the NLEEs. Obtained results are in adherence with the results obtained by [25] under few limitations. Three dimensional profiles for each solution obtained and their corresponding contour plots are presented clearly.

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